(5.1)

Many concepts in probability, physics, and everyday life have a surprising connection with the spatial concept of area and volume. For example, the amount of money that you earned last week is identical to the area of a rectangle whose length is the number of hours worked.

For a more interesting example, consider the following problem:

Ex. Two forgetful profs. want to meet in order to write an exam together. Since they cannot remember the exact time of metting, they agree to do the following:

Each will come anywhere between 1:00 pm and 2:00 pm, wait for the other for 15 mins $\equiv 1/4$ hr, and leave if the other prof. doesn't show up by then.

what is the probability that the meeting takes place?

Solution: Let x represent the time in hours after 1:00 pm when Prof. 1 shows up for the meeting. Similarly, let y be the corresponding time for Prof. 2. Then $x \in [0,1]$, $y \in [0,1]$ and (x, y) represents the event that Prof. 1 comes x^{th} of an hour past 1:00 and Prof. 2 comes y^{th} of an hour past 1:00. For instance, $(1/4, 1/2)$ means that Prof. 1 arrives at 1:15 pm and Prof. 2 arrives at 1:30 pm.

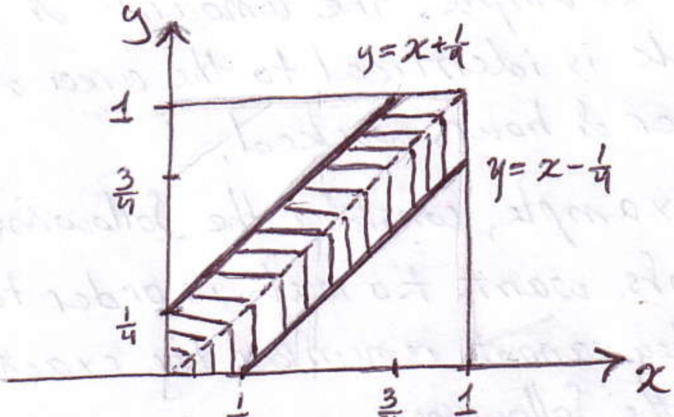
Clearly the set $S = \{(x,y) : x \in [0,1], y \in [0,1]\}$ is the event space (i.e. the collection of all possible outcomes). Geometrically, S is the unit square.

Let M be the event that the two Profs. meet. Then $(x,y) \in M$ iff

$$x - 1/4 \leq y \leq x + 1/4$$

because, if Prof. 2 arrives more that 1/4 hour earlier than Prof. 1, Prof 2. wouldn't wait for his colegue. Similarly, Prof 2 cannot arrive

$\frac{1}{4}$ hour later than Prof. 1, because Prof 1 wouldn't wait for him. It follows that M is the shaded region in the drawing below.

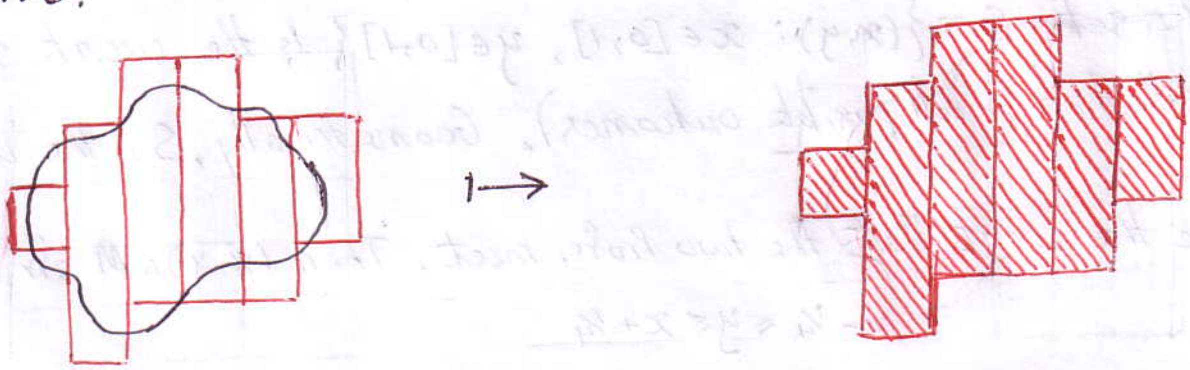


The likelihood that the two profs. meet is therefore $\frac{A(M)}{A(S)} = A(M)$ where $A(M)$ is the area of the region M. It is easy to verify that $A(M) = \frac{7}{16} \equiv 43.75\%$, (why?)

In this section we recall a few techniques for evaluating area and volume.

Intuition behind Area & Volume

Intuitively, area is a measurement of the "amount of space" enclosed by a curve. When this curve is continuous, we may use the principle of "pixelization" to approximate the area outlined by the curve.



This approximation tends as a limit to the exact area as the pixels are getting smaller in width.

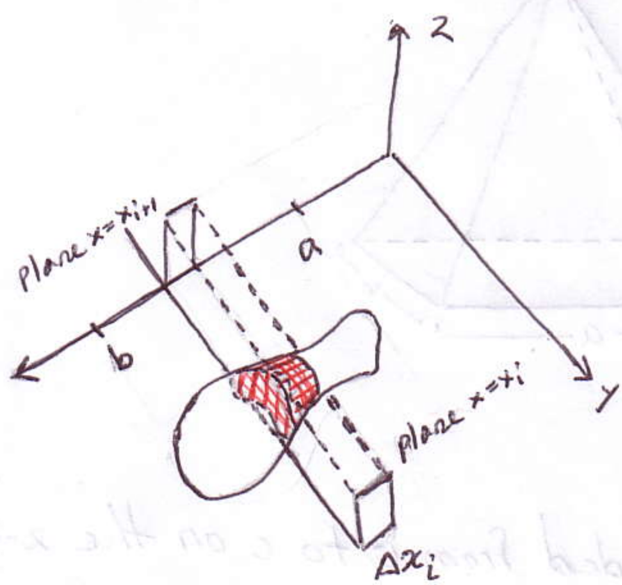
Similarly, volume is a measurement of the 'amount of space' enclosed by a surface. The volume of the space bounded within the surface may be approximated by the sum of the volumes of cylindrical segments, provided that the areas of the cross-sections of the solids are known. This method for computing volumes is known as Cavalieri's principle.

Cavalieri's principle

Suppose we have a solid whose length along the x-axis is $b-a$, where a is the position of the solid's tail and b is the position of the head of the solid. Let $A(x)$ denote its cross-sectional area in a plane P_x, $x \in [a,b]$. Then the volume of this solid is given by $\int_a^b A(x)\,dx$.

To see that this is so, partition the interval $[a,b]$ into n subintervals:

$$a = x_0 < x_1 < \ldots < x_n < b,$$

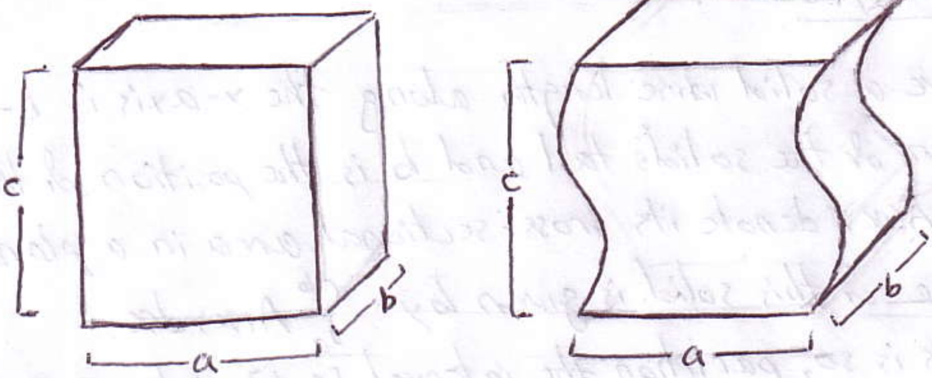


Observe that the piece of solid wedged between the planes $P_{x_{i+1}}$ and P_{x_i} is roughly a cylinder with cross-sectional area $A(x_i)$ and width $\Delta x_i = x_{i+1} - x_i$. Since the volume of this cylindrical section is $A(x_i)\Delta x_i$, it follows that the volume of the solid, $V(S)$, is approximately the Riemann sum $\sum_{i=0}^{n-1} A(x_i)\Delta x_i$.

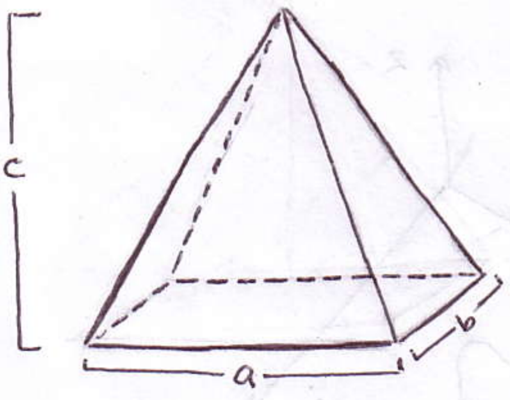
Therefore $V(S) = \lim_{n\to\infty} \sum_{i=0}^{n-1} A(x_i)\Delta x_i = \int_a^b A(x)\,dx$.

Ex. Find the volume of the shapes below

a)



b)



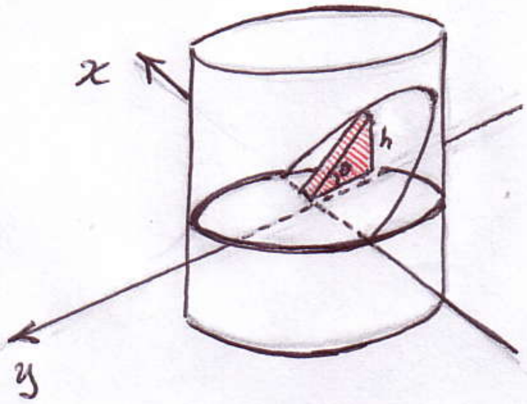
Solution: Both solids are bounded from 0 to c on the z-axis.

a) The cross-sections $A(z) = ab$. Thus the volumes are $\int_0^c ab\,dz = abc$.

b) You should verify that the intersection of the pyramid and P_z, $z \in [0,c]$ is a rectangle with dimensions $\frac{a}{2c}(c-z)$ and $\frac{b}{2c}(c-z)$. Thus $A(z) = \frac{ab}{4c^2}(c-z)^2$. The volume of the pyramid is therefore

$$\int_0^c \frac{ab}{4c^2}(c-z)^2 dz = \frac{abc}{12}$$

Ex. (Marsden & Tromba): A lumberjack cuts out a wedge-shaped piece W of a cylindrical tree of radius r obtained by making two saw cuts to the tree's center, one horizontally and one at an angle θ. Compute the volume of the wedge W.



Solution: The area of the triangular cross-section at x, $A(x)$, is given by $A(x) = \frac{1}{2} y h$ where h and y satisfy $y \geq 0$, $h \geq 0$, $\frac{h}{y} = \tan\theta$, and $x^2+y^2=r^2$. Therefore $h = y\tan\theta$ and $y = \sqrt{r^2-x^2}$. Consequently $A(x) =$
$= \frac{1}{2} y h = \frac{1}{2}(r^2-x^2)\tan\theta$ and $Vol(W) = \int_{-r}^{r} A(x)\,dx =$
$$= \int_{-r}^{r} \frac{r^2-x^2}{2}\tan\theta\, dx = \frac{\tan\theta}{2}\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{2}{3}r^3\tan\theta$$